Scientific Programming: Part B

Lecture 2

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[credits: thanks to Prof. Alberto Montresor]

Introduction

Goal: ***estimate the complexity in time of algorithms***

- Definitions
- Computing models
- Evaluation examples
- Notation

Why?

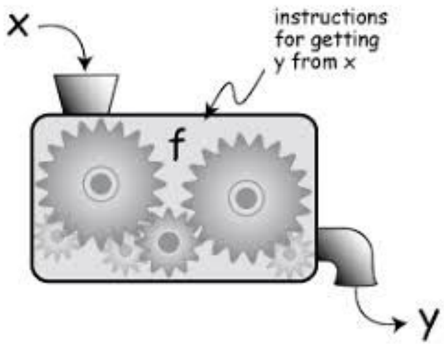
- To estimate the time needed to process a given input
- To estimate the largest input computable in a reasonable time
- To compare the efficiency of different algorithms
- To optimize the most important part

Complexity

The **complexity** of an algorithm can be defined as a **function mapping the size of the input** to the **time** required to get the result.

We need to define:

1. How to measure the **size of the input**
2. How to measure **time**



How to measure the size of inputs

Uniform cost model

- *The input size is equal to the number of elements composing it*
- Example: minimum search in a list of n elements

In some cases (e.g. factorial of a number) we need to consider how many bits we use to represent inputs

Logarithmic cost model

- *The input size is equal to the number of bits representing it*
- Example: binary number multiplication of numbers of n bits

In several cases...

- We can assume that the *elements* are represented by a constant number of bits
- The two measures are the same, apart from a constant multiplication factor

Measuring time is trickier...

Time ≡ wall-clock time

The actual time used to complete an algorithm

It depends on too many parameters:

- how good is the programmer
- programming language
- code generated by the compiler/interpreter
- CPU, memory, hard-disk, etc.
- operating system, other processes currently running, etc.

We need a more abstract representation of time

Random Access Model (RAM): time

Let's count the **number of basic operations**

What are basic operations?

Time ≡ number of basic instructions

An instruction is considered basic if it can be executed in constant time by the processor

Basic

- `a = a*2` ? Yes (unless numbers have arbitrary precision)
- `math.cos(d)` ? Yes
- `min(A)` ? No (modern GPUs are highly parallel and can be constant)

Example: minimum

Let's count the **number of basic operations for min.**

- Each statement requires a constant time to be executed (even len???)
- This constant may be different for each statement
- Each statement is executed a given number of times, function of n (size of input).

```
def my_faster_min(S):
    min_so_far = S[0] #first element
    i = 1
    while i < len(S):
        if S[i] < min_so_far:
            min_so_far = S[i]
        i = i +1
    return min_so_far
```

Example: minimum

Let's count the **number of basic operations for min.**

- Each statement requires a constant time to be executed (even len???)
- This constant may be different for each statement
- Each statement is executed a given number of times, function of n (size of input).

Code	Cost	Number of times
`def my_faster_min(S):`		
`    min_so_far = S[0] #first element`	c1	1
`    i = 1`	c2	1
`    while i < len(S):`	c3	n
`        if S[i] < min_so_far:`	c4	n-1
`            min_so_far = S[i]`	c5	n-1 (worst case)
`        i = i +1`	c6	n-1
`    return min_so_far`	c7	1

T(n) = c1 + c2 + c3*n + c4*(n-1) + c5*(n-1)+c6*(n-1)+c7

= (c3+c4+c5+c6)*n + (c1+c2-c4-c5-c6+c7) = **a*n + b**

Example: lookup

Let's count the **number of basic operations for lookup.**

- The list is split in two parts: left size ⌊(n-1)/2⌋ right size ⌊n/2⌋

```
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)
```

Example: lookup

Let's count the **number of basic operations for lookup.**

- The list is split in two parts: left size $\lfloor (n-1)/2 \rfloor$ right size $\lfloor n/2 \rfloor$

	Cost	**Executed?**	
def lookup_rec(L, v, start,end):		end < start	end ≥ start
if end < start:	c1	1	1
return -1	c2	1	0
else:			
m = (start + end)//2	c3	0	1
if L[m] == v: *#found!*	c4	0	1
return m	c5	0	0 (worst case)
elif v < L[m]: *#look to the left*	c6	0	1
return lookup_rec(L, v, start, m-1)	c7 + **T(⌊(n-1)/2⌋)**	0	0/1
else: *#look to the right*			
return lookup_rec(L, v, m+1, end)	c7+ **T(⌊n/2⌋)**	0	1/0

Note: lookup_rec is not a basic operation!!!

Lookup: recurrence relation

Assumptions:

- For simplicity, n is a power of 2: n = 2^k
- The searched element is not present (worst case)
- At each call, we select the right part whose size is n/2 (instead of (n-1)/2)

if start > end (n=0):

$$T(n) = c_1 + c_2 = c$$

if start ⩽ end (n>0):

$$T(n) = T(n/2) + c_1 + c_3 + c_4 + c_6 + c_7 = T(n/2) + d$$

Recurrence relation:

$$T(n) = \begin{cases} c & n = 0 \\ T(n/2) + d & n \geq 1 \end{cases}$$

Lookup: recurrence relation

$$T(n) = \begin{cases} c & n = 0 \\ T(n/2) + d & n \geq 1 \end{cases}$$

Solution

Remember that: $n = 2^k \Rightarrow k = log_2 n$

$$\begin{aligned} T(n) &= T(n/2) + d \\ &= (T(n/4) + d) + d = T(n/4) + 2d \\ &= (T(n/8) + d) + 2d = T(n/8) + 3d \\ &\dots \\ &= T(1) + kd \\ &= T(0) + (k+1)d \\ &= kd + (c + d) \\ &= d \log n + e. \end{aligned}$$

as seen before, the complexity is logarithmic
Note: in computer science log is log2.

Asymptotic notation

Complexity functions → "big-Oh" notation (omicron)

So far...

- Lookup: $T(n) = d \cdot logn + e$ logarithmic O(log n)
- Minimum: $T(n) = a \cdot n + b$ linear O(n)
- Naive Minimum: $T(n) = f \cdot n^2 + g \cdot n + h$ quadratic O(n^2)

we ignore the "less impacting" parts (like constants or n in naive, ...) and focus on the predominant ones

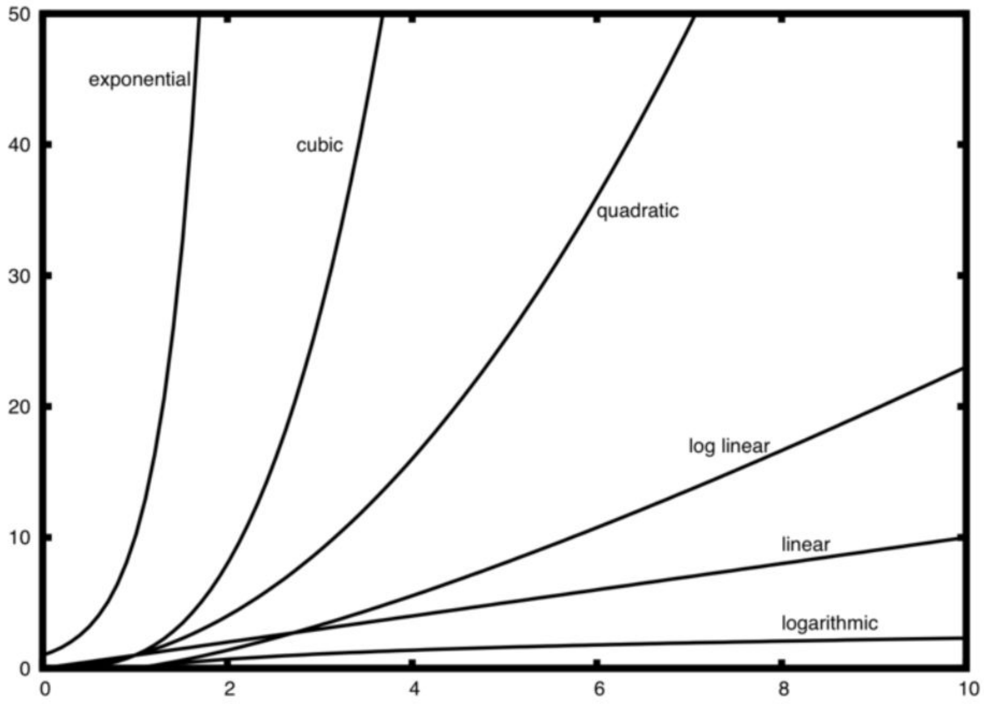
Asymptotic notation

Complexity classes

$f(n)$	$n = 10^1$	$n = 10^2$	$n = 10^3$	$n = 10^4$	**Type**
$\log n$	3	6	9	13	logarithmic
$\sqrt{n}$	3	10	31	100	sublinear
n	10	100	1000	10000	linear
$n \log n$	30	664	9965	132877	log-linear
n^2	10^2	10^4	10^6	10^8	quadratic
n^3	10^3	10^6	10^9	10^{12}	cubic
2^n	1024	10^{30}	10^{300}	10^{3000}	exponential

Note: these are “trends” (we hide all constants that might have an impact for small inputs). For small inputs exponential algorithms might still be acceptable (especially if nothing better exists!)

Asymptotic notation



[Miller, Ranum, Problem solving with Algorithms and Data structures]

O,Ω,Θ notations

Definition – O notation

Let $g(n)$ be a cost function; $O(g(n))$ is the set of all functions $f(n)$ such that:

$$\exists c > 0, \exists m \geq 0 : f(n) \leq cg(n), \forall n \geq m$$

- How we read it: $f(n)$ is "big-Oh" of $g(n)$
- How we write it: $f(n) = O(g(n))$
- $g(n)$ is asymptotic upper bound for $f(n)$
- $f(n)$ grows at most as $g(n)$

O, Ω, Θ notations

Definition – Ω notation

Let $g(n)$ be a cost function; $\Omega(g(n))$ is the set of all functions $f(n)$ such that:

$$\exists c > 0, \exists m \geq 0 : f(n) \geq cg(n), \forall n \geq m$$

- How we read it: $f(n)$ is "big-omega" of $g(n)$
- How we write it: $f(n) = \Omega(g(n))$
- $g(n)$ is an asymptotic lower bound for $f(n)$
- $f(n)$ grows at least as $g(n)$

O,Ω,Θ notations

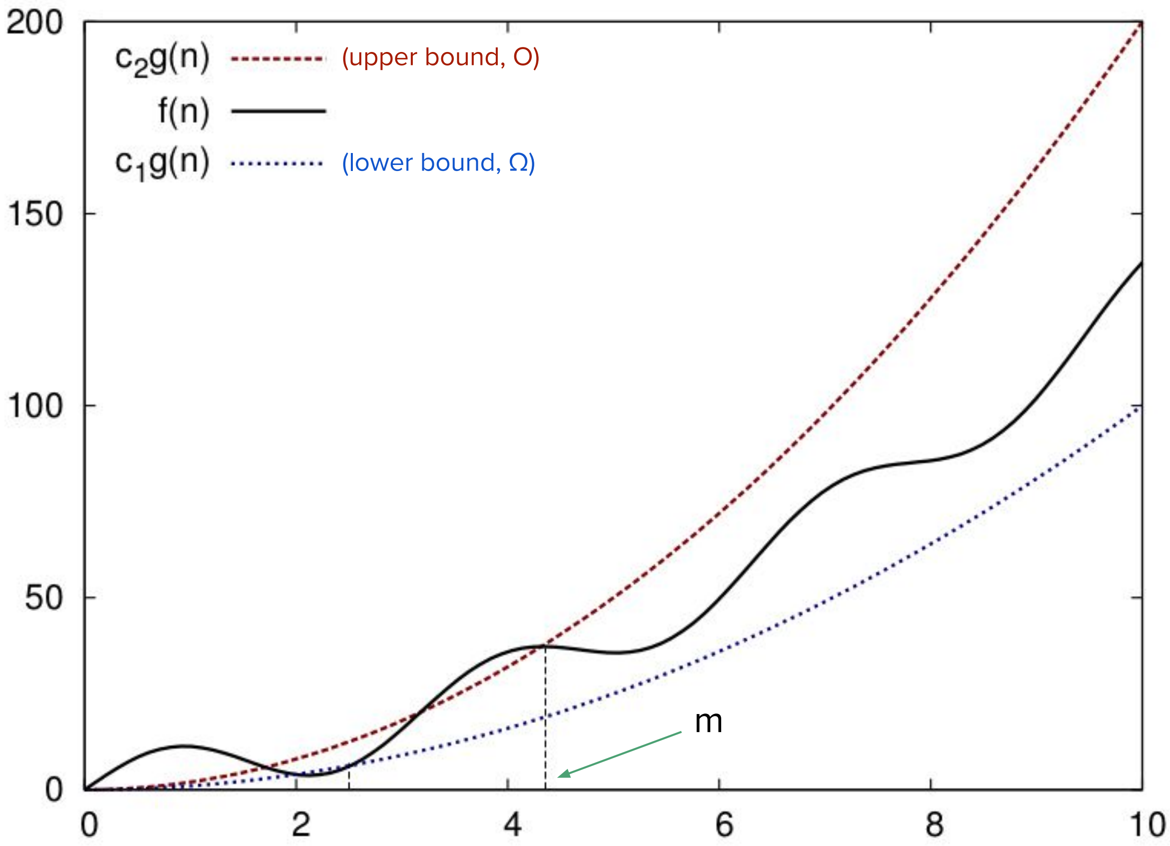
Definition – Notation Θ

Let $g(n)$ be a cost function; $\Theta(g(n))$ is the set of all functions $f(n)$ such that:

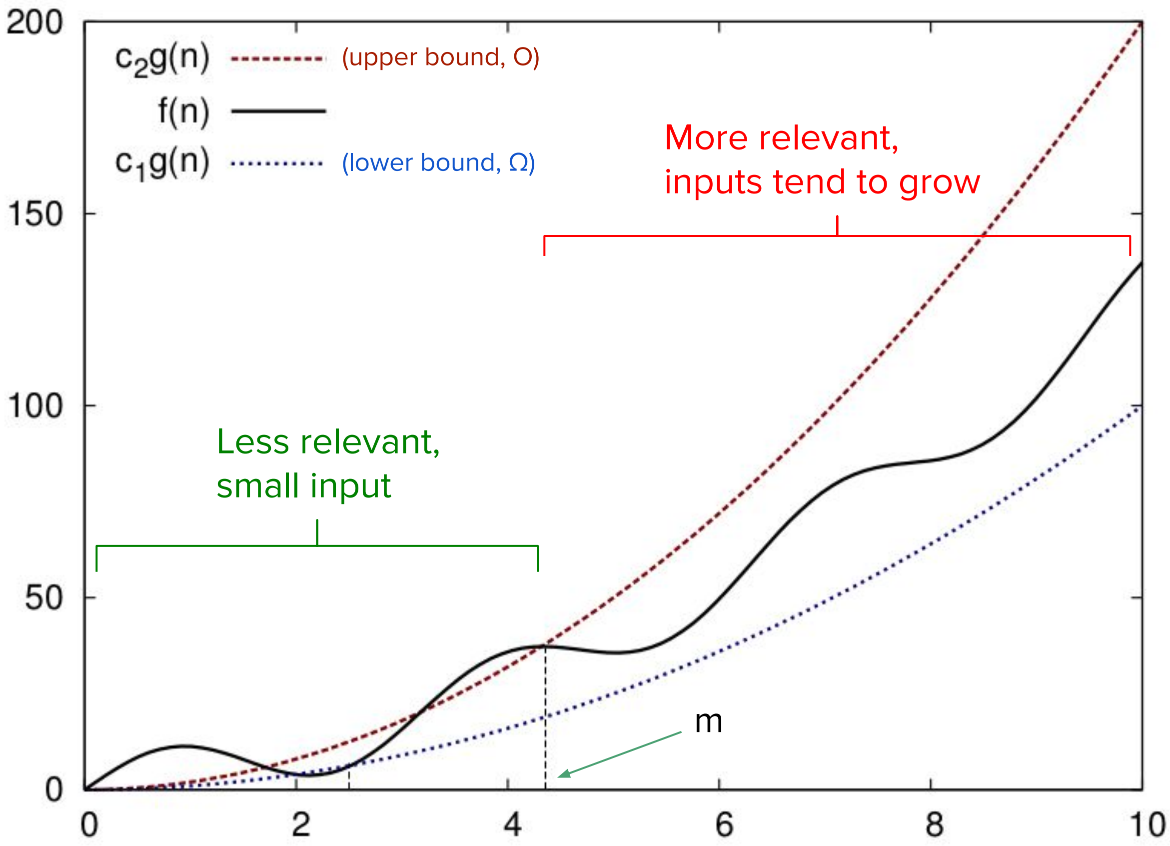
$$\exists c_1 > 0, \exists c_2 > 0, \exists m \geq 0 : c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq m$$

- How we read it: $f(n)$ is "theta" of $g(n)$
- How we write it: $f(n) = \Theta(g(n))$
- $f(n)$ grows as $g(n)$
- $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

O,Ω,Θ notations



O,Ω,Θ notations



Exercise: True or False?

$$f(n) = 10n^3 + 2n^2 + 7 \stackrel{?}{=} O(n^3)$$

We need to prove that (i.e. find a c and m such that):

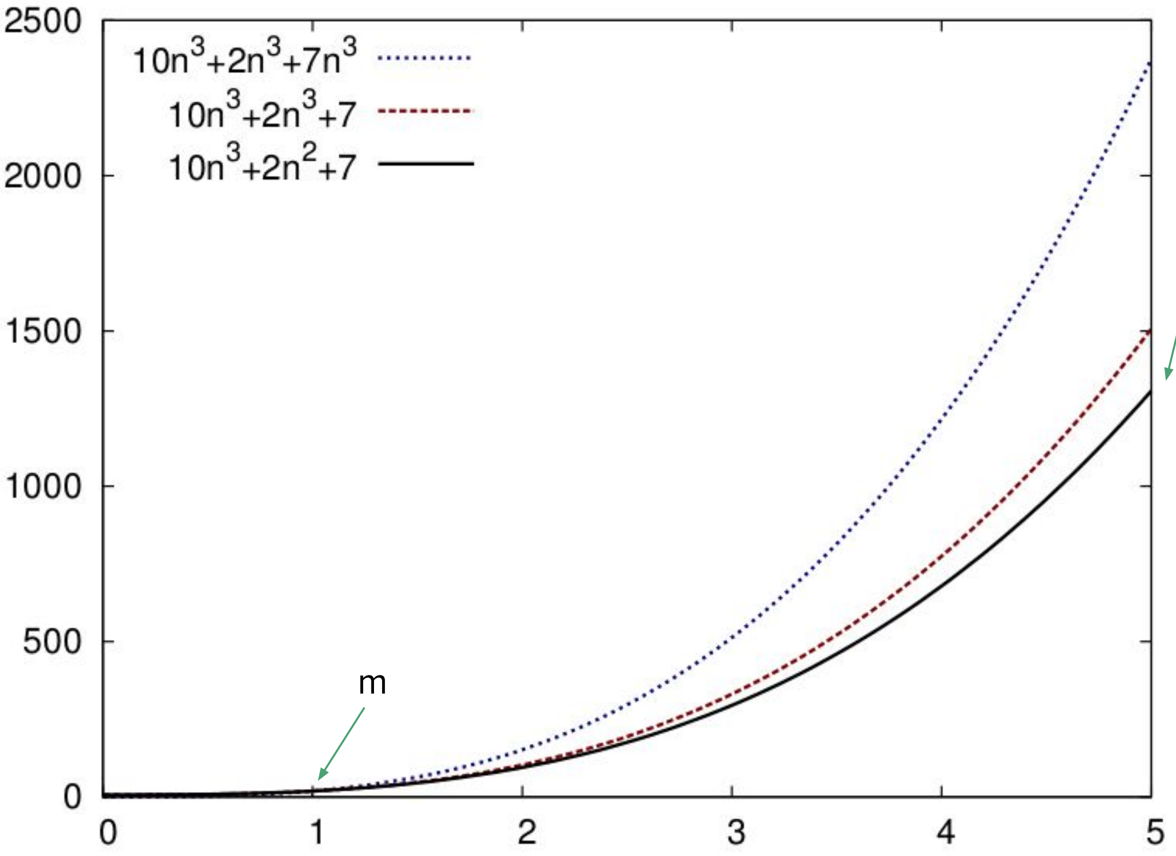
$$\exists c > 0, \exists m \geq 0 : f(n) \leq c \cdot n^3, \forall n \geq m$$

$$\begin{aligned} f(n) &= 10n^3 + 2n^2 + 7 & \\ &\leq 10n^3 + 2n^3 + 7 & \forall n \geq 0 \\ &\leq 10n^3 + 2n^3 + 7n^3 & \forall n \geq 1 \\ &= 19n^3 & \\ &\stackrel{?}{\leq} cn^3 & \end{aligned}$$

which is true for each $c \geq 19$ and for each $n \geq 1$, thus $m = 1$.

In graphical terms

$$f(n) = 10n^3 + 2n^2 + 7$$



Exercise: True or False?

$$f(n) = 3n^2 + 7n \stackrel{?}{=} \Theta(n^2)$$

We need to prove that (i.e. find a c and m such that):

$$\exists c_1 > 0, \exists m_1 \geq 0 : f(n) \geq c_1 \cdot n^2, \forall n \geq m_1$$

lower bound (Ω)

and that

$$\exists c_2 > 0, \exists m_2 \geq 0 : f(n) \leq c_2 \cdot n^2, \forall n \geq m_2$$

upper bound (O)

Exercise: True or False?

$$f(n) = 3n^2 + 7n \stackrel{?}{=} \Theta(n^2)$$

We need to prove that (i.e. find a c and m such that):

$$\exists c_1 > 0, \exists m_1 \geq 0 : f(n) \geq c_1 \cdot n^2, \forall n \geq m_1$$ **lower bound (Ω)**

$$\begin{aligned} f(n) &= 3n^2 + 7n \\ &\geq 3n^2 && n \geq 0 \\ &\stackrel{?}{\geq} c_1 n^2 \end{aligned}$$

which is true for each $c_1 \leq 3$ and for each $n \geq 0$, thus $m_1 = 0$ → f(n) = Ω(n^2)

Exercise: True or False?

$$f(n) = 3n^2 + 7n \stackrel{?}{=} \Theta(n^2)$$

We need to prove that (i.e. find a c and m such that):

$$\exists c_2 > 0, \exists m_2 \geq 0 : f(n) \leq c_2 \cdot n^2, \forall n \geq m_2$$

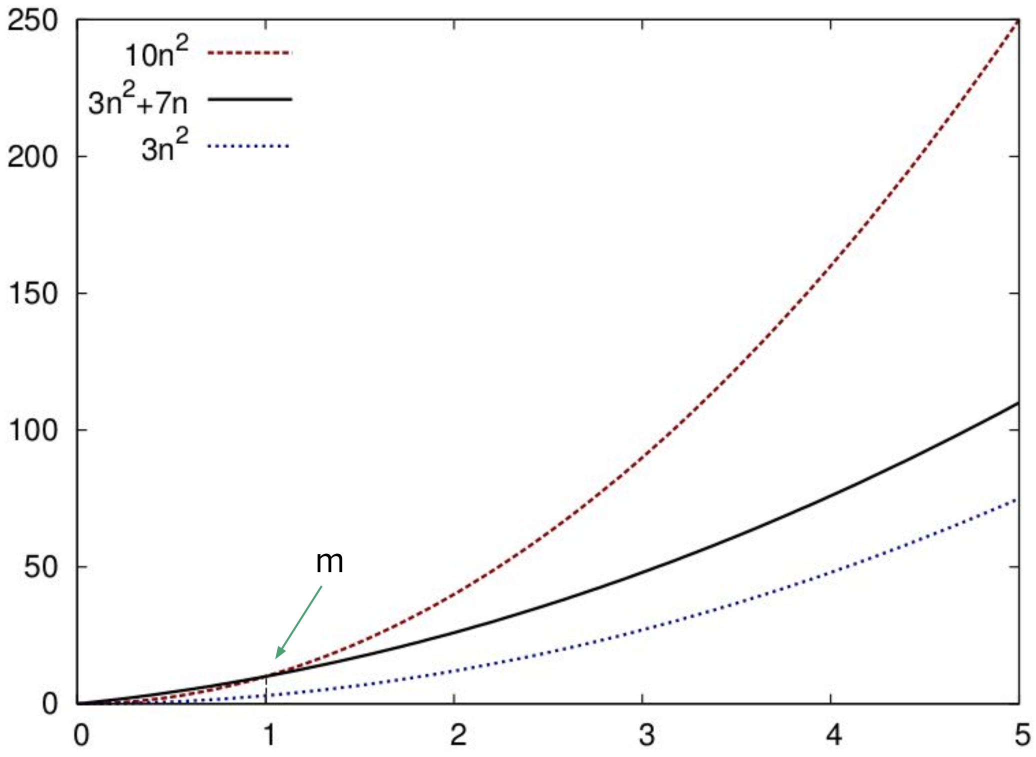
upper bound (O)

$$\begin{aligned} f(n) &= 3n^2 + 7n \\ &\leq 3n^2 + 7n^2 && n \geq 1 \\ &= 10n^2 \\ &\stackrel{?}{\leq} c_2 n^2 \end{aligned}$$

which is true for each $c_2 \geq 10$ and for all $n \geq 1$, hence $m_2 = 1$. ⇒ f(n) = O(n^2)

⇒ $f(n) = 3n^2 + 7n = \Theta(n^2)$

In graphical terms: 3n^2+7n is Θ(n^2)



True or False?

$$n^2 \stackrel{?}{=} O(n)$$

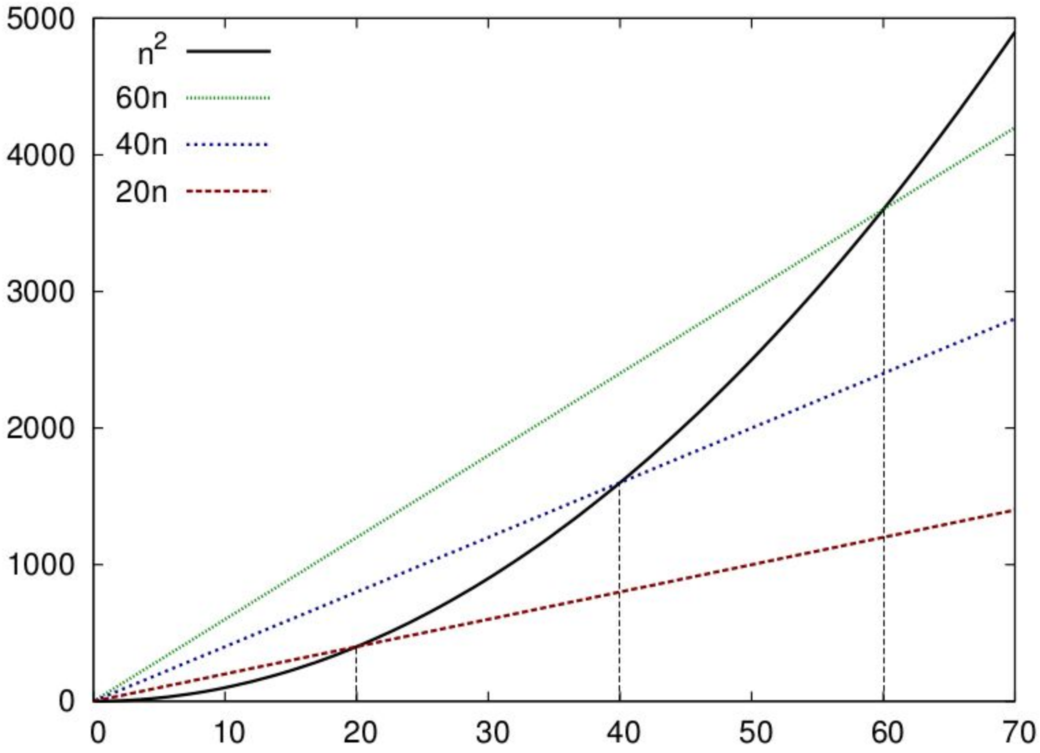
We want to prove that $\exists c > 0, \exists m > 0 : n^2 \leq cn, \forall n \geq m$

- We get this: $n^2 \leq cn \Leftrightarrow c \geq n$
- This means that c should grow with n, i.e. we cannot choose a constant c valid for all $n \geq m$

$$\Rightarrow \quad n^2 \neq O(n)$$

True or False?

$$n^2 \neq O(n)$$



Exercise:

$$n^2 = O(n^3)$$

we cannot find a constant C making n grow faster than n^2

Properties

Polynomial expressions

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots a_1 n + a_0, a_k > 0 \Rightarrow f(n) = \Theta(n^k)$$

Constant elimination

$$f(n) = O(g(n)) \Leftrightarrow af(n) = O(g(n)), \forall a > 0$$
$$f(n) = \Omega(g(n)) \Leftrightarrow af(n) = \Omega(g(n)), \forall a > 0$$

Meaning:

- We only care about the highest degree of the polynomial
- Multiplicative constants, do not change the asymptotic complexity (e.g. constants costs due to language, technical implementation,...)

Properties

Sums

$$f_1(n) = O(g_1(n)), f_2(n) = O(g_2(n)) \Rightarrow$$
$$f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$$
$$f_1(n) = \Omega(g_1(n)), f_2(n) = \Omega(g_2(n)) \Rightarrow$$
$$f_1(n) + f_2(n) = \Omega(\min(g_1(n), g_2(n)))$$

Relation with algorithm analysis

- If an algorithm is composed by two parts, one which is $\Theta(n^2)$ and one which $\Theta(n)$, the resulting complexity is $\Theta(n^2 + n) = \Theta(n^2)$

We only care about the "computationally more expensive" part to solve of the algorithm.

$$O(n \cdot logn + n) = O(n \cdot logn)$$

Properties

Products

$$f_1(n) = O(g_1(n)), f_2(n) = O(g_2(n)) \Rightarrow f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$$
$$f_1(n) = \Omega(g_1(n)), f_2(n) = \Omega(g_2(n)) \Rightarrow f_1(n) \cdot f_2(n) = \Omega(g_1(n) \cdot g_2(n))$$

Relation with algorithm analysis

- If algorithm A calls algorithm B n times, and the complexity of algorithm B is $\Theta(n \log n)$, the resulting complexity is $\Theta(n^2 \log n)$.

```
for i in range(n):
        call_to_function_that_is_n^2_log_n()
```

$\Theta(n^2 \log n)$

Classification

Is it possible to create a total order between the main function classes.

For each $0 < r < s, 0 < h < k, 1 < a < b$:

$$O(1) \subset O(\log^r n) \subset O(\log^s n) \subset O(n^h) \subset O(n^h \log^r n) \subset$$
$$O(n^h \log^s n) \subset O(n^k) \subset O(a^n) \subset O(b^n)$$

Examples:

$$O(logn) \subset O(\sqrt[3]{n}) \subset O(\sqrt{n})$$

$$O(2^{n+1}) = O(2 \cdot 2^n) = O(2^n)$$

No matter the exponent, **(log n)^r** will always be better than **n**)...
Same thing for **n log n** vs **n** etc...

Complexity of maxsum: Θ(n^3)

```
def max_sum_v1(A):
    max_so_far = 0
    N = len(A)
    for i in range(N):
        for j in range(i,N):
            tmp_sum = sum (A[i:j+1])
            max_so_far = max(tmp_sum, max_so_far)

    return max_so_far
```

Intuitively:

we perform two loops of length N one into the other ➔ cost N^2

sum is not a basic operation (cost N):

➔ overall cost N^3

The complexity of this algorithm can be approximated as follows (we are counting the number of sums that are executed).

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (j - i + 1)$$

We want to prove that $T(n) = \theta(n^3)$, i.e.

$$\exists c_1, c_2 > 0, \exists m \geq 0 : c_1 n^3 \leq T(n) \leq c_2 n^3, \forall n \geq m$$

Complexity of maxsum: O(n^3)

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}(j-i+1) \\
&\le \sum_{i=0}^{n-1}\sum_{j=i}^{n-1} n \\
&\le \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} n \\
&= \sum_{i=0}^{n-1} n^2 \\
&\le n^3 \le c_2 n^3
\end{aligned}
$$

This inequality is true for $n \ge m = 0$ and $c_2 \ge 1$.

➡ O(n^3)

Complexity of maxsum: Ω(n^3)

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}(j-i+1) \\
&\geq \sum_{i=0}^{n/2}\sum_{j=i}^{i+n/2-1}(j-i+1) \\
&\geq \sum_{i=0}^{n/2}\sum_{j=i}^{i+n/2-1} n/2 \\
&= \sum_{i=0}^{n/2} n^2/4 \geq n^3/8 \geq c_1 n^3
\end{aligned}
$$

This inequality is true for $n \geq m = 0$ and $c_1 \leq$ 1/8 ➡ Ω(n^3)

➡ Θ(n^3)

Complexity of maxsum -version 2: Ω(n^2)

```
def max_sum_v2(A):
    N = len(A)
    max_so_far = 0

    for i in range(N):
        tot = 0 #ACCUMULATOR!
        for j in range(i,N):
            tot = tot + A[j]
            max_so_far = max(max_so_far, tot)
    return max_so_far
```

The complexity of this algorithm can be approximated as follows (we are counting the number of sums that are executed).

$$T(n) = \sum_{i=0}^{n-1} n - i$$

Complexity of maxsum -version 2: Θ(n^2)

We want to prove that $T(n) = \theta(n^2)$.

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{n-1} n - i \\
&= \sum_{i=1}^{n} i \\
&= \frac{n(n+1)}{2} = \Theta(n^2)
\end{aligned}
$$

Gauss

This does not require further proofs.

Complexity of maxsum -version 4: Θ(n)

```
def max_sum_v4(A):
    max_so_far = 0 #Max found so far
    max_here = 0   #Max slice ending at cur pos

    for i in range(len(A)):
        max_here = max(A[i] + max_here, 0)
        max_so_far = max(max_so_far, max_here)
    return max_so_far
```

This is rather easy!
Constant operations (sum and max of 2 numbers) performed n times

Complexity is Θ**(n)**

Complexity of maxsum -version 3

```
from itertools import accumulate

def max_sum_v3_rec_bis(A,i,j):
    if i == j:
        return max(0,A[i])
    m = (i+j)//2
    maxL = max_sum_v3_rec_bis(A,i,m)
    maxR = max_sum_v3_rec_bis(A, m+1, j)
    maxML = max(accumulate(A[m:-len(A) + i -1: -1]))
    maxMR = max(accumulate(A[m+1:j+1]))
    return max(maxL, maxR, maxML+ maxMR)

def max_sum_v3(A):
    return max_sum_v3_rec_bis(A,0,len(A) - 1)
```

Recursive algorithm, recurrence relation

Bear with me a minute. We will get back to this later...!

Recurrences

Recurrence equations

Whenever the complexity of a recursive algorithm is computed, this is expressed through recurrence equation, i.e. a mathematical formula defined in a... recursive way!

Example

$$T(n) = \begin{cases} 2T(n/2) + n & n > 1 \\ \Theta(1) & n \leq 1 \end{cases}$$

Recurrences

Closed formulas

Our goal is to obtain, whenever possible, a closed formula that represents the complexity class of our function.

Example

$$T(n) = \Theta(n \log n)$$

Master Theorem

Theorem

Let a and b two integer constants such that $a \geq 1$ e $b \geq 2$, and let c, β be two real constants such that $c > 0$ e $\beta \geq 0$. Let $T(n)$ be defined by the following recurrence:

$$T(n) = \begin{cases} aT(n/b) + cn^\beta & n > 1 \\ \Theta(1) & n \leq 1 \end{cases}$$

Given $\alpha = \log a / \log b = \log_b a$, then:

$$T(n) = \begin{cases} \Theta(n^\alpha) & \alpha > \beta \\ \Theta(n^\alpha \log n) & \alpha = \beta \\ \Theta(n^\beta) & \alpha < \beta \end{cases}$$

Note: the schema covers cases when input of size **n** is split in **b** sub-problems, to get the solution the algorithm is applied recursively **a** times. **c**$\mathbf{n^\beta}$ is the cost of the algorithm after the recursive steps.

Examples

Theorem

Let a and b two integer constants such that $a \geq 1$ e $b \geq 2$, and let c, β be two real constants such that $c > 0$ e $\beta \geq 0$. Let $T(n)$ be defined by the following recurrence:

$$T(n) = \begin{cases} aT(n/b) + cn^{\beta} & n > 1 \\ \Theta(1) & n \leq 1 \end{cases}$$

Given $\alpha = \log a / \log b = \log_b a$, then:

$$T(n) = \begin{cases} \Theta(n^{\alpha}) & \alpha > \beta \\ \Theta(n^{\alpha} \log n) & \alpha = \beta \\ \Theta(n^{\beta}) & \alpha < \beta \end{cases}$$

Algo: splits the input in two, applies the procedure recursively 4 times and has a linear cost to assemble the solution at the end.

Recurrence	a	b	$\log_b a$	Case	Function
$T(n) = 4T(n/2) + n$	4	2	2	(1)	$T(n) = \Theta(n^2)$
$T(n) = 3T(n/2) + n$	3	2	$\log_2 3$	(1)	$T(n) = \Theta(n^{\log_2 3})$
$T(n) = 2T(n/2) + n$	2	2	1	(2)	$T(n) = \Theta(n \log n)$
$T(n) = T(n/2) + 1$	1	2	0	(2)	$T(n) = \Theta(\log n)$
$T(n) = 9T(n/3) + n^3$	9	3	2	(3)	$T(n) = \Theta(n^3)$

n^1.58

Note: the schema covers cases when input of size **n** is split in **b** sub-problems, to get the solution the algorithm is applied recursively **a** times. **cn$^{\beta}$** is the cost of the algorithm after the recursive steps.

maxsum - version 3

```
from itertools import accumulate

def max_sum_v3_rec_bis(A,i,j):
    if i == j:
        return max(0,A[i])
    m = (i+j)//2
    maxL = max_sum_v3_rec_bis(A,i,m)
    maxR = max_sum_v3_rec_bis(A, m+1, j)
    maxML = max(accumulate(A[m:-len(A) + i -1: -1]))
    maxMR = max(accumulate(A[m+1:j+1]))
    return max(maxL, maxR, maxML+ maxMR)

def max_sum_v3(A):
    return max_sum_v3_rec_bis(A,0,len(A) - 1)
```

For this, we need to define a recurrence relation:

$$T(n) = 2T(n/2) + cn$$

The algorithm **splits the input in two** “equally-sized” sub-problems (**m = i+j//2**) and **applies itself recursively 2 times**.
The accumulate after the recursive part **is linear cn**.

maxsum - version 3

```python
from itertools import accumulate

def max_sum_v3_rec_bis(A,i,j):
    if i == j:
        return max(0,A[i])
    m = (i+j)//2
    maxL = max_sum_v3_rec_bis(A,i,m)
    maxR = max_sum_v3_rec_bis(A, m+1, j)
    maxML = max(accumulate(A[m:-len(A) + i -1: -1]))
    maxMR = max(accumulate(A[m+1:j+1]))
    return max(maxL, maxR, maxML+ maxMR)

def max_sum_v3(A):
    return max_sum_v3_rec_bis(A,0,len(A) - 1)
```

Theorem

Let a and b two integer constants such that $a \geq 1$ e $b \geq 2$, and let c, β be two real constants such that $c > 0$ e $\beta \geq 0$. Let $T(n)$ be defined by the following recurrence:

$$T(n) = \begin{cases} aT(n/b) + cn^\beta & n > 1 \\ \Theta(1) & n \leq 1 \end{cases}$$

Given $\alpha = \log a / \log b = \log_b a$, then:

$$T(n) = \begin{cases} \Theta(n^\alpha) & \alpha > \beta \\ \Theta(n^\alpha \log n) & \alpha = \beta \\ \Theta(n^\beta) & \alpha < \beta \end{cases}$$

For this, we need to define a recurrence relation:

$$T(n) = 2T(n/2) + cn$$

$$\alpha = \log_2 2 = 1 \text{ and } \beta = 1$$

$$T(n) = \Theta(n \log n)$$